



## Enhancing Rheumatic Fever Analysis via Tritopological Approximation Spaces for Data Reduction

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### Abstract

This paper introduces the concept of tritopological approximation space, extending conventional approximation space by drawing upon topological spaces and precisely defined binary relations within a universe of discourse. Through meticulous construction of subbases, this progressive paradigm shift facilitates a comprehensive analysis of rough sets within the domain of tritopological approximation spaces. Additionally, the study pioneer's multiple membership functions and inclusion functions, enhancing the analytical framework and enabling more effective re-definition of rough approximations. To illustrate the practical advantages, real-life application examples are presented, focusing on the implementation of data reduction methods within the context of rheumatic fever—a prevalent disease characterized by diverse symptoms among patients, despite a consistent diagnosis. This research contributes to the advancement of rough set theory and its applications in addressing complex, real-world problems.

**Keywords:** rough sets; tritopological spaces; rheumatic fever; decision-making.

# 1 Introduction

In recent years, rough set theory and its extensions have increasingly captivated scholars, particularly in the fields of computer science and artificial intelligence. Pawlak's groundbreaking work in 1982 marked the inception of rough set theory [21], where he utilized an equivalence relation for classification purposes. Over time, Pawlak's rough set theory has evolved into a vital extension of set theory, providing robust tools for the analysis of diverse data types. As a result, researchers have proposed various generalizations, utilizing either arbitrary or specific relations. For instance, these include similarity relations [3], general relations [30], coverings [7, 11, 16], topological approaches such as topological covering-based rough sets [16], topological soft rough sets [6, 8], nano-ordered topological spaces [25], and fuzzy decision-making [1, 2, 28]. Moreover, there are several applications using neighborhood systems [4], soft rough sets [8, 17], fuzzy set applications [5, 18, 24], and other applications utilizing fuzzy systems [27, 30].

Its practical applications extend across various domains, including artificial intelligence disciplines such as machine learning, pattern recognition, decision analysis, cognitive sciences, and intelligent decision-making [17]. Within this framework, the representation of uncertain data relies on a pair of certain sets, namely the lower and upper approximations. These, defined in terms of equivalence classes, constitute fundamental concepts in rough set theory. The lower approximation encompasses equivalence classes entirely contained within the set, while the upper approximation includes all equivalence classes with a nonempty intersection with the set. Topological structures [13] have been utilized in many application fields such as medical applications [8, 10], economic applications [9], fuzzy applications [24], and rough set properties [19, 22, 28].

In this paper, we introduce a novel approach that explores tritopological methods to address vagueness in a more versatile manner. The primary focus is on addressing the boundary region, a critical aspect of any vague concept. Our proposed models effectively reduce the boundary region of the concept of interest by employing binary relations defined according to the attributes present in an information system. This strategy aims to create three distinct topological spaces, forming the basis for introducing a sequence of comprehensive generalizations. Firstly, we introduce and investigate some new generalized rough approximations based on the notion of tritopological approximation space. These represent accurate tools to measure the roughness and exactness of sets, which are very useful in decision making and medical applications. On the other hand, we prove that the proposed approaches are generalizations of the previous works in the literature. Some results proving the properties of these methods are introduced and investigated with counterexamples. Moreover, the relationships among them and other techniques are presented and examined. The introduced methods will be used for an important application, namely, in the topological reduction of enhancing rheumatic fever analysis. We believe that the suggested approaches will open the way for more topological directions in the line of tritopological spaces in many fields such as fuzzy sets, soft sets, and fuzzy soft set theory.

The structure of the paper is prepared as follows:

- In Section 2, a comprehensive overview is presented on generalized rough sets induced by binary relations, along with the introduction of the concepts of tri-near open sets.
- Section 3 establishes the concept of a tritopological approximation space, shedding light on significant accuracy metrics intricately associated with this construct.
- In Section 4, a thorough exploration of noteworthy properties within tritopological spaces is undertaken, accompanied by the introduction of the concept of rough sets within this specific contextual framework.

- Section 5 outlines an application approach aimed at effecting data reduction within multi-valued information systems.
- Lastly, Section 6 serves to conclude this paper by providing a summarization of our research findings and suggesting potential directions for future research endeavors.

## 2 Preliminaries

The underpinning of rough set theory can be traced back to the need for representing subsets of a universal set through equivalence classes derived from a partition. This partition establishes a topological space referred to as the approximation space  $A = (U, R)$ , where  $U$  denotes the universe set, and  $R$  represents the equivalence relation [21]. The equivalence classes derived from  $R$  are widely acknowledged as granules, or elementary sets. Each element  $x \in U$  is associated with its respective equivalence class denoted as  $[x]_R$ . These equivalence classes serve a pivotal role in defining rough approximations. To comprehend the fundamental principles of rough set theory, we will reference the key concepts outlined in [21] in the subsequent discussion.

**Definition 2.1.** [21] *If  $U$  is the universe and  $R$  is an equivalence relation on  $U$ , then  $(U, R)$  is called Pawlak approximation space. The lower and upper approximations of a subset  $X \subseteq U$  are defined as  $\underline{R}(X) = \{x \in U : [x]_R \subseteq X\}$  and  $\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \phi\}$ , respectively. Also consider  $POS_R(X) = \underline{R}(X)$ ,  $NEG_R = U - \overline{R}(X)$ , and  $BN_R = \overline{R}(X) - \underline{R}(X)$  represent the positive, the negative, and the boundary region of  $X$ , respectively.*

**Definition 2.2.** [21] *Assumed an equivalence relation  $R$  on the set  $U$  and a subset  $X \subseteq U$ , Pawlak established a method to ascertain the degree of exactness of  $X$  using the accuracy measure:*

$$\alpha_R(X) = \frac{|\underline{R}(X)|}{|\overline{R}(X)|}, \text{ where } \overline{R}(X) \neq \phi.$$

**Definition 2.3.** [21] *Consider an equivalence relation  $R$  on the universe  $U$ , and let  $X$  be a subset of  $U$ . For any element  $x$  in  $X$ , we define rough membership relations to  $X$  as follows:  $x \in X$  iff  $x \in \underline{R}(X)$  and  $x \in A$  iff  $x \in \overline{R}(X)$ .*

**Definition 2.4.** [12, 14, 29] *In the context of an equivalence relation  $R$  on the set  $U$ , the rough inclusion relations for every  $X$  and  $Y$  subsets of  $U$  can be formulated as follows:*

$$(X \subset \sim Y) \text{ if } \underline{R}(X) \subseteq \underline{R}(Y), \text{ and } (A \subset \sim B) \text{ if } \overline{R}(X) \subseteq \overline{R}(Y).$$

**Definition 2.5.** [12, 14, 29] *Assumed an equivalence relation  $R$  on the set  $U$  and  $X \subseteq U$ . Then:*

1. If  $\underline{R}(X) \neq \phi$  and  $\overline{R}(X) \neq U$ , then  $X$  is termed roughly  $R$ -definable.
2. If  $\underline{R}(X) = \phi$  and  $\overline{R}(X) \neq U$ , then  $X$  is termed internally  $R$ -undefinable.
3. If  $\underline{R}(X) \neq \phi$  and  $\overline{R}(X) = U$ , then  $X$  is termed externally  $R$ -undefinable.
4. If  $\underline{R}(X) = \phi$  and  $\overline{R}(X) = U$ , then  $X$  is termed totally  $R$ -undefinable.

**Remark 2.1.** *The initial extension of rough set approximations rooted in topological frameworks was introduced by [29] and termed as topological rough approximations. In this method, the conventional lower and upper approximations are substituted with the interior and closure operators, respectively. These operators are defined as follows:  $int(A) = \cup\{G \subseteq U : G \in \tau, G \subseteq A\}$  and  $cl(A) = \cap\{F \subseteq U : U - F \in \tau, A \subseteq F\}$ .*

**Definition 2.6.** [14] Let  $\tau_1, \tau_2,$  and  $\tau_3$  be three topologies on  $U$ . The set  $U$  together with these three topologies is called a tritopological space and is denoted by  $(U, \tau_1, \tau_2, \tau_3)$ .

**Definition 2.7.** [12] A subset  $A$  of a tritopological space  $(U, \tau_1, \tau_2, \tau_3)$  is called a tri-open set if  $A \in \tau_1 \cup \tau_2 \cup \tau_3$  and the complement of tri-open set is tri-closed.

**Definition 2.8.** [12] Let  $(U, \tau_1, \tau_2, \tau_3)$  be a tritopological space and  $A \subseteq U$ . The tri-closure and tri-interior of  $A$  (briefly,  $tri-cl(A)$  and  $tri-int(A)$ ) are defined, respectively, as follows:

1.  $tri-cl(A) = \cap\{F : F \supseteq A \text{ and } F \text{ is tri-closed}\}.$
2.  $tri-int(A) = \cup\{G : G \subseteq A \text{ and } G \text{ is tri-open}\}.$

In the following, we list some tri-near open sets.

**Definition 2.9.** Consider  $(U, \tau_1, \tau_2, \tau_3)$  is a tritopological space, a subset  $A \subseteq U$  is said to be:

1. tri-semi open [26] if  $A \subseteq tri-cl(tri-int(A)).$
2. tri-preopen [26] if  $A \subseteq tri-int(tri-cl(A)).$
3. tri- $\alpha$ -open [20] if  $A \subseteq tri-int(tri-cl(tri-int(A)).$
4. tri-b-open [23] if  $A \subseteq [tri-int(tri-cl(A))] \cup [tri-cl(tri-int(A))].$
5. tri- $\beta$ -open [20] if  $A \subseteq tri-cl(tri-int(tri-cl(A))).$
6. tri- $\delta\beta$ -open [20] if  $A \subseteq tri-cl(tri-int(tri-cl_\delta(A))).$

The family of all tri-semi open (respectively, tri-preopen, tri- $\alpha$ -open, tri-b-open, tri- $\beta$ -open, and tri- $\delta\beta$ -open) sets in a tritopological space  $(U, \tau_1, \tau_2, \tau_3)$  is denoted by  $tri-SO(X)$  (respectively,  $tri-PO(X)$ ,  $tri-\alpha O(X)$ ,  $tri-bO(X)$ ,  $tri-\beta O(X)$ , and  $tri-\delta\beta O(X)$ ).

### 3 Tritopological Approximation Spaces

In this section, we commence with the introduction and thorough exploration of the concepts of tri-lower approximation and tri-upper approximation, delving into a comprehensive analysis of their inherent properties. By harnessing tritopological bases derived from binary relations, we showcase the ability to construct tri-lower and tri-upper approximation operators.

**Definition 3.1.** Consider a binary relation, denoted as  $R$ , defined on a universal set  $U$ . For any elements  $x$  and  $y$  belonging to  $U$ , we define the set  $R_r(x)$  as the collection of elements  $y$  for which the relation  $xRy$  holds true, that is, when the pair  $(x, y)$  is an element of the relation  $R$ . This collection is referred to as the right neighborhood of the element  $x$ . The aggregate of all such right neighborhoods, denoted as  $S_r$ , encompasses sets of this nature for all elements  $x$  in  $U$ . This collection  $S_r$  forms a foundational subset for constructing a topology, designated as  $\tau_r$ , on the universal set.

Similarly, by considering the reverse of the binary relation, we define the left neighborhood  $R_\ell(x)$  as the set of elements  $y$  for which  $yRx$  holds true, that is, when the pair  $(y, x)$  lies within the relation  $R$ . This set represents the left neighborhood of the element  $x$ . Analogously, the family  $S_\ell$ ,

comprised of these left neighborhoods for all elements  $x$  in  $U$ , constitutes a fundamental subset for establishing another topology, denoted as  $\tau_\ell$ , on the universal set.

Combining these notions, we introduce the combined family  $S_{r\ell}$ , formed by the union of  $S_r$  and  $S_\ell$ . This collective set forms a foundational subset for constructing a third topology, designated as  $\tau_{r\ell}$ , on the universal set  $U$ . In summary, the defined binary relation  $R$  on the universal set  $U$  gives rise to three distinct topologies:  $\tau_r$  based on right neighborhoods,  $\tau_\ell$  based on left neighborhoods, and  $\tau_{r\ell}$  based on the amalgamation of both right and left neighborhoods.

The approximation space denoted as  $Tri \mathcal{T} = (U, R, \tau_r, \tau_\ell, \tau_{r\ell})$  represents an extended form of the traditional approximation space introduced by Pawlak. This augmented version is referred to as a tritopological approximation space due to its incorporation of three distinct topologies:  $\tau_r$ ,  $\tau_\ell$ , and  $\tau_{r\ell}$ . This advancement provides a more comprehensive framework for handling approximation and generalization within the context of the original Pawlak approximation space.

**Definition 3.2.** The subbases  $S_r$ ,  $S_\ell$ , and  $S_{r\ell}$  delineate a lower (resp. upper) approximation for  $A \subseteq U$  in the following ways:

For  $S_r$

- 1)  $\underline{S}_r(A) = A \cap (\cup \{R_r(x) \in S_r : R_r(x) \subseteq A\})$ ,
- 2)  $\overline{S}_r(A) = A \cup (\cap \{R_r(x) \in S_r : R_r(x) \cap A \neq \phi\})$ .

For  $S_\ell$

- 1)  $\underline{S}_\ell(A) = A \cap (\cup \{R_\ell(x) \in S_\ell : R_\ell(x) \subseteq A\})$ ,
- 2)  $\overline{S}_\ell(A) = A \cup (\cap \{R_\ell(x) \in S_\ell : R_\ell(x) \cap A \neq \phi\})$ .

For  $S_{r\ell}$

- 1)  $\underline{S}_{r\ell}(A) = A \cap (\cup \{M \in S_{r\ell} : M \subseteq A\})$ ,
- 2)  $\overline{S}_{r\ell}(A) = A \cup (\cap \{M \in S_{r\ell} : M \cap A \neq \phi\})$ .

Therefore, the accuracy measures of the above approximations of  $A$  are given, respectively, as:

$$\mu_{S_r}(A) = \frac{|\underline{S}_r(A)|}{|\overline{S}_r(A)|}, \mu_{S_\ell}(A) = \frac{|\underline{S}_\ell(A)|}{|\overline{S}_\ell(A)|} \text{ and } \mu_{S_{r\ell}}(A) = \frac{|\underline{S}_{r\ell}(A)|}{|\overline{S}_{r\ell}(A)|}.$$

**Definition 3.3.** A subset  $A \subseteq U$  of the tritopological approximation space  $(U, R, \tau_r, \tau_\ell, \tau_{r\ell})$  is called tri-open if  $A \in Tri \mathcal{T} = \tau_r \cup \tau_\ell \cup \tau_{r\ell}$  and the complement of a tri-open set is a tri-closed set.

**Definition 3.4.** A subset  $A$  of tritopological approximation space  $(U, R, \tau_r, \tau_\ell, \tau_{r\ell})$  is called tri- $\beta$ -open if  $A \subseteq tri-cl(tri-int(tri-cl(A)))$ .

**Definition 3.5.** A subset  $A$  of  $(U, R, \tau_r, \tau_\ell, \tau_{r\ell})$  is called tri- $\delta$ -closed if  $A = \overline{S}_\delta(A)$ , where  $\overline{S}_\delta(A) = \{x \in U : A \cap tri-int(tri-cl(G)) \neq \emptyset, G \in tri-\tau \text{ and } x \in G\}$ . The complement of a tri- $\delta$ -closed set is tri- $\delta$ -open.

**Definition 3.6.** In a tritopological approximation space  $(U, R, \tau_r, \tau_\ell, \tau_{r\ell})$ , a subset  $A$  of  $U$  is deemed tri- $\delta\beta$ -open if it satisfies the condition  $A \subseteq tri-cl(tri-int(\overline{S}_\delta(A)))$ . The complement of a tri- $\delta\beta$ -open set is referred to as tri- $\delta\beta$ -closed. The family of all tri- $\delta\beta$ -open sets is denoted as  $tri-\delta\beta O(U)$ , and the set of tri- $\delta\beta$ -closed sets is denoted as  $tri-\delta\beta C(U)$ .

**Definition 3.7.** In a tritopological approximation space  $(U, R, \tau_r, \tau_\ell, \tau_{r\ell})$ , the tri-lower approximation, tri-upper approximation, tri-boundary, tri-positive region, and tri-negative region, as well as the tri-accuracy of the approximations of a subset  $A$  are given, respectively, as follows:

$$\begin{aligned} \underline{Tri T}(A) &= \cup \{G \in tri O(U) : G \subseteq A\} = \text{tri-interior of } A. \\ \overline{Tri T}(A) &= \cap \{F \in tri C(U) : A \subseteq F\} = \text{tri-closure of } A. \\ Tri T BON(A) &= \overline{Tri T}(A) - \underline{Tri T}(A). \\ Tri T POS(A) &= \underline{Tri T}(A). \\ Tri T NEG(A) &= U - \overline{Tri T}(A). \\ Tri T \mu(A) &= \frac{|\underline{Tri T}(A)|}{|\overline{Tri T}(A)|}, \text{ where } |\overline{Tri T}(A)| \neq 0. \end{aligned}$$

**Definition 3.8.** In a tritopological approximation space  $(U, R, \tau_r, \tau_\ell, \tau_{r\ell})$ , where  $A$  is a subset of  $U$ . Then,  $\forall j \in \{p, s, \gamma, \alpha, \beta, \delta\}$  the tri- $j$ -lower, tri- $j$ -upper approximations, tri- $j$ -boundary, tri- $j$ -positive and tri- $j$ -negative regions, tri- $j$ -accuracy of the approximations of  $A$  are defined respectively by:

$$\begin{aligned} \underline{Tri T}_j(A) &= \cup \{G \in tri j O(U) : G \subseteq A\} = \text{tri}_j\text{-interior of } A. \\ \overline{Tri T}_j(A) &= \cap \{F \in tri j C(U) : A \subseteq F\} = \text{tri}_j\text{-closure of } A. \\ Tri T BON_j(A) &= \overline{Tri T}_j(A) - \underline{Tri T}_j(A). \\ Tri T POS_j(A) &= \underline{Tri T}_j(A). \\ Tri T NEG_j(A) &= U - \overline{Tri T}_j(A). \\ Tri T \mu_j(A) &= \frac{|\underline{Tri T}_j(A)|}{|\overline{Tri T}_j(A)|}, \text{ where } |\overline{Tri T}_j(A)| \neq 0. \end{aligned}$$

**Example 3.1.** Suppose  $R$  represents a binary relation on  $U = \{a, b, c, d, e\}$  defined by  $R = \{(a, a), (a, e), (b, c), (b, d), (c, e), (d, a), (d, e), (e, e)\}$ . Hence, the subbase of  $\tau_r$  is  $S_r = \{\{a, e\}, \{c, d\}, \{e\}\}$ , the subbase of  $\tau_\ell$  is  $S_\ell = \{\{a, d\}, \{b\}, \{a, c, d, e\}\}$  and the subbase of  $\tau_{r\ell}$  is  $S_{r\ell} = S_r \cup S_\ell = \{\{b\}, \{e\}, \{a, d\}, \{a, e\}, \{c, d\}, \{a, c, d, e\}\}$ . Then  $\tau_r = \{U, \phi, \{e\}, \{a, e\}, \{c, d\}, \{c, d, e\}, \{a, c, d, e\}\}$ ,  $\tau_\ell = \{U, \phi, \{b\}, \{a, d\}, \{a, b, d\}, \{a, c, d, e\}\}$  and  $\tau_{r\ell} = \{U, \phi, \{a\}, \{b\}, \{d\}, \{e\}, \{a, b\}, \{a, d\}, \{a, e\}, \{b, d\}, \{b, e\}, \{c, d\}, \{d, e\}, \{a, b, d\}, \{a, b, e\}, \{a, c, d\}, \{a, d, e\}, \{b, c, d\}, \{b, d, e\}, \{c, d, e\}, \{a, b, c, d\}, \{a, b, d, e\}, \{a, c, d, e\}, \{b, c, d, e\}\}$ . Table 1 illustrates the tri-accuracy measures  $Tri T \mu(A)$ ,  $Tri T \mu_\beta(A)$  and  $Tri T \mu_{\delta\beta}(A)$  for specific subsets of the universe.

Table 1: Tri-accuracy measure  $Tri T \mu(A)$ ,  $Tri T \mu_\beta(A)$  and  $Tri T \mu_{\delta\beta}(A)$  of some subsets.

$A \subseteq U$	$Tri T \mu(A)$	$Tri T \mu_\beta(A)$	$Tri T \mu_{\delta\beta}(A)$
$\{d\}$	1/2	1/2	1
$\{b, d\}$	2/3	2/3	1
$\{c, d\}$	1	1	1
$\{a, b, d\}$	3/4	3/4	1
$\{a, c, e\}$	2/3	2/3	1
$\{b, c, d\}$	1	1	1
$\{a, b, c, d\}$	1	1	1
$\{a, b, c, e\}$	3/4	3/4	1
$\{a, b, d, e\}$	4/5	4/5	1

Utilizing the information provided in Table 1, it becomes apparent that the precision degree for the subset  $A = \{a, c, e\}$  is 67% when using the tri-accuracy measure, 67% when using the tri- $\beta$ -accuracy measure,

and 100% when using the  $\text{tri-}\delta\beta$ -accuracy measure. Thus, the  $\text{tri-}\delta\beta$ -accuracy measure appears to be the most effective among the three measures.

Considering the above approximations, any subset  $A \subseteq U$  can be divided into the following regions:

1.  $\underline{\text{Tri Edg}}(A) = A - \underline{\text{Tri T}}(A)$  named “tri-internal edge” of  $A$ .
2.  $\underline{\text{Tri Edg}}_j(A) = A - \underline{\text{Tri T}}_j(A)$  named “tri  $j$ -internal edge” of  $A$ .
3.  $\overline{\text{Tri Edg}}(A) = \overline{\text{Tri T}}(A) - A$  named “tri-external edge” of  $A$ .
4.  $\overline{\text{Tri Edg}}_j(A) = \overline{\text{Tri T}}_j(A) - A$  named “tri  $j$ -external edge” of  $A$ .
5.  $\text{Tri T BON}(A) = \overline{\text{Tri T}}(A) - \underline{\text{Tri T}}(A)$  named “tri-boundary” of  $A$ .
6.  $\text{Tri T BON}_j(A) = \overline{\text{Tri T}}_j(A) - \underline{\text{Tri T}}_j(A)$  named “tri  $j$ -boundary” of  $A$ .
7.  $\text{Tri T NEG}(A) = U - \overline{\text{Tri T}}(A)$  named “tri-negative” of  $A$ .
8.  $\text{Tri T NEG}_j(A) = U - \overline{\text{Tri T}}_j(A)$  named “tri  $j$ -negative” of  $A$ .

**Proposition 3.1.** For any tritopological approximation space  $\text{Tri T} = (U, R, \tau_r, \tau_\ell, \tau_{r\ell})$ , and  $A \subseteq U$ :

1.  $\text{Tri T BON}(A) = \underline{\text{Tri Edg}}(A) \cup \overline{\text{Tri Edg}}(A)$ .
2.  $\text{Tri T BON}_j(A) = \underline{\text{Tri Edg}}_j(A) \cup \overline{\text{Tri Edg}}_j(A)$ .

*Proof.*

1.
 
$$\begin{aligned} \text{Tri T BON}(A) &= \overline{\text{Tri T}}(A) - \underline{\text{Tri T}}(A) \\ &= (\overline{\text{Tri T}}(A) - A) \cup (A - \underline{\text{Tri T}}(A)) \\ &= \overline{\text{Tri Edg}}(A) \cup \underline{\text{Tri Edg}}(A). \end{aligned}$$

2. The proof will follow the same way as in number 1.

□

Consider the tritopological approximation space denoted as  $\text{Tri T} = (U, R, \tau_r, \tau_\ell, \tau_{r\ell})$  and let  $A \subseteq U$ . We now proceed to establish the subsequent membership functions:

**Definition 3.9.** Consider the tritopological approximation space denoted as  $\text{Tri T} = (U, R, \tau_r, \tau_\ell, \tau_{r\ell})$  and let  $A \subseteq U$ , we proceed to formally generate the subsequent membership functions:

1.  $x \in_{\text{tri-open}} A$  iff  $x \in \underline{\text{Tri T}}(A)$ , which is named as Tri – strong membership function.
2.  $x \in_{\text{tri-j-open}} A$  iff  $x \in \underline{\text{Tri T}}_j(A)$ , which is named as Tri–  $j$ – strong membership function.
3.  $x \in_{\overline{\text{tri-open}}} A$  iff  $x \in \overline{\text{Tri T}}(A)$ , which is named as Tri – weak membership function.
4.  $x \in_{\overline{\text{tri-j-open}}} A$  iff  $x \in \overline{\text{Tri T}}_j(A)$ , which is named as Tri–  $j$ – weak membership function.

**Remark 3.1.** Referring to the previously defined membership function explanations, we can now reformulate the tri-lower (or tri-upper) approximation for  $A$  is a subset of  $U$  by the below items:

1.  $\underline{Tri T}(A) = \{x \in A : x \in_{tri-open} A\}$ .
2.  $\underline{Tri T}_j(A) = \{x \in A : x \in_{tri-j-open} A\}$ .
3.  $\overline{Tri T}(A) = \{x \in A : x \in_{tri-open} A\}$ .
4.  $\overline{Tri T}_j(A) = \{x \in A : x \in_{tri-j-open} A\}$ .

**Remark 3.2.** Let  $(U, R, \tau_r, \tau_\ell, \tau_{r\ell})$  be a tritopological approximation space and  $A \subseteq U$ . Then:

$$x \in_{tri-open} A \implies x \in_{tri-j-open} A. \tag{1}$$

$$x \in_{tri-j-open} A \implies x \in_{tri-open} A. \tag{2}$$

**Definition 3.10.** Consider the tritopological approximation space denoted as  $(U, R, \tau_r, \tau_\ell, \tau_{r\ell})$ , where  $A$  and  $B$  are subsets of  $U$ . In this context, we characterize  $A$  and  $B$  as:

1. If  $\underline{Tri T}(A) = \underline{Tri T}(B)$ , then they are tri-roughly bottom equal (symbolized by  $A \approx_{tri-open} B$ ).
2. If  $\underline{Tri T}_j(A) = \underline{Tri T}_j(B)$ , then they are tri- $j$ -roughly bottom equal (symbolized by  $A \approx_{tri-j-open} B$ ).
3. If  $\overline{Tri T}(A) = \overline{Tri T}(B)$ , then they are tri-roughly top equal (symbolized by  $A \simeq_{tri-open} B$ ).
4. If  $\overline{Tri T}_j(A) = \overline{Tri T}_j(B)$ , then they are tri- $j$ -roughly top equal (symbolized by  $A \simeq_{tri-j-open} B$ ).

**Definition 3.11.** Given the tritopological approximation space defined as  $(U, R, \tau_r, \tau_\ell, \tau_{r\ell})$ , where  $A$  and  $B$  are subsets of  $U$ , we proceed to define:

1.  $A$  is Tri-roughly bottom included in  $B$  ( $AC_{\sim}^{tri-open} B$ ) if  $\underline{Tri T}(A) \subseteq \underline{Tri T}(B)$ .
2.  $A$  is Tri -  $j$ -roughly bottom included in  $B$  ( $AC_{\sim}^{tri-j-open} B$ ) if  $\underline{Tri T}_j(A) \subseteq \underline{Tri T}_j(B)$ .
3.  $A$  is Tri-roughly top included in  $B$  ( $AC_{\sim_{tri-open}} B$ ) if  $\overline{Tri T}(A) \subseteq \overline{Tri T}(B)$ .
4.  $A$  is Tri -  $j$ -roughly top included in  $B$  ( $AC_{\sim_{tri-j-open}} B$ ) if  $\overline{Tri T}_j(A) \subseteq \overline{Tri T}_j(B)$ .
5.  $A$  is Tri-roughly included in  $B$  ( $AC_{\sim_{tri-open}} B$ ) if ( $AC_{\sim}^{tri-open} B$ ) and ( $AC_{\sim_{tri-open}} B$ ).
6.  $A$  is Tri -  $j$ -roughly included in  $B$  ( $AC_{\sim_{tri-j-open}} B$ ) if ( $AC_{\sim}^{tri-j-open} B$ ) and ( $AC_{\sim_{tri-j-open}} B$ ).

## 4 Characteristics of Tritopological Rough Approximations

In this section, we present essential characteristics of tritopological spaces and lay the foundation for understanding rough sets within the framework of  $Tri T = (U, R, \tau_r, \tau_\ell, \tau_{r\ell})$ .

**Definition 4.1.** Given the tritopological approximation space defined as  $(U, R, \tau_r, \tau_\ell, \tau_{r\ell})$ , where  $A \subseteq U$ :

1. If  $\underline{Tri T}(A) = \overline{Tri T}(A)$  (i.e.  $Tri T BON(A) = \emptyset$ ), then  $A$  is called *tri-definable set*.
2. If  $\underline{Tri T}_j(A) = \overline{Tri T}_j(A)$  (i.e.  $Tri T BON_j(A) = \emptyset$ ), then  $A$  is called *tri-j-definable set*.
3. If  $\underline{Tri T}(A) \neq \overline{Tri T}(A)$  (i.e.  $Tri T BON(A) \neq \emptyset$ ), then  $A$  is called *tri-rough set*.
4. If  $\underline{Tri T}_j(A) \neq \overline{Tri T}_j(A)$  (i.e.  $Tri T BON_j(A) \neq \emptyset$ ), then  $A$  is called *tri-j-rough set*.

**Definition 4.2.** In the context of the tritopological approximation space denoted as  $(U, R, \tau_r, \tau_\ell, \tau_{r\ell})$ , where  $A \subseteq U$ :

1. A set  $A$  is termed *roughly tri-definable* if both  $\underline{Tri T}(A) \neq \emptyset$  and  $\overline{Tri T}(A) \neq U$ .
2. A set  $A$  is labeled *roughly tri-j-definable* if both  $\underline{Tri T}_j(A) \neq \emptyset$  and  $\overline{Tri T}_j(A) \neq U$ .
3. A set  $A$  is termed *internally tri-undefinable* if both  $\underline{Tri T}(A) = \emptyset$  and  $\overline{Tri T}(A) \neq U$ .
4. A set  $A$  is referred to as *internally tri-j-undefinable* if both  $\underline{Tri T}_j(A) = \emptyset$  and  $\overline{Tri T}_j(A) \neq U$ .
5. A set  $A$  is termed *externally tri-undefinable* if both  $\underline{Tri T}(A) \neq \emptyset$  and  $\overline{Tri T}(A) = U$ .
6. A set  $A$  is referred to as *externally tri-j-undefinable* if both  $\underline{Tri T}_j(A) \neq \emptyset$  and  $\overline{Tri T}_j(A) = U$ .
7. A set  $A$  is termed *totally tri-undefinable* if both  $\underline{Tri T}(A) = \emptyset$  and  $\overline{Tri T}(A) = U$ .
8. A set  $A$  is referred to as *totally tri-j-undefinable* if both  $\underline{Tri T}_j(A) = \emptyset$  and  $\overline{Tri T}_j(A) = U$ .

**Remark 4.1.** For any tritopological approximation space  $(U, R, \tau_r, \tau_\ell, \tau_{r\ell})$ . The following hold:

1.  $Tri - \delta\beta RD(U) \supseteq Tri - \beta RD(U) \supseteq Tri - RD(U)$ .
2.  $Tri - \delta\beta IUD(U) \subseteq Tri - \beta IUD(U) \subseteq Tri - IUD(U)$ .
3.  $Tri - \delta\beta EUD(U) \subseteq Tri - \beta EUD(U) \subseteq Tri - EUD(U)$ .
4.  $Tri - \delta\beta TUD(U) \subseteq Tri - \beta TUD(U) \subseteq Tri - TUD(U)$ .

**Proposition 4.1.** For any tritopological approximation space  $(U, R, \tau_r, \tau_\ell, \tau_{r\ell})$  and for all  $x, y \in U$ , we have:

1. If  $x \in \overline{Tri T}(\{y\})$  and  $y \in \overline{Tri T}(\{x\})$ , then  $\overline{Tri T}(\{x\}) = \overline{Tri T}(\{y\})$ .
2. If  $x \in \overline{Tri T}_j(\{y\})$  and  $y \in \overline{Tri T}_j(\{x\})$ , then  $\overline{Tri T}_j(\{x\}) = \overline{Tri T}_j(\{y\})$ .

*Proof.*

1. Since  $Tri - cl(\{y\})$  is a  $Tri - closed$  set containing  $x$  while  $Tri - cl(\{x\})$  is the smallest  $Tri - closed$  set containing  $x$ , thus  $Tri - cl(\{x\}) \subseteq Tri - cl(\{y\})$ . Hence,  $\overline{Tri T}(\{x\}) \subseteq \overline{Tri T}(\{y\})$ . The opposite inclusion follows by symmetry  $Tri - cl(\{y\}) \subseteq Tri - cl(\{x\})$ . Therefore,  $\overline{Tri T}(\{y\}) \subseteq \overline{Tri T}(\{x\})$ .
2. The proof for part 2 follows the same approach as in part 1.

□

## 5 Topological Data Reduction for the Analysis of Rheumatic Fever

This application focuses on implementing practical data reduction techniques within the context of rheumatic fever, a prevalent ailment characterized by a variety of symptoms that vary across patients, despite consistent diagnoses. Our dataset includes information from five patients aged between 9 and 12 years, all of whom experienced the onset of arthritis between ages 3 and 5. Rheumatic fever is known for its wide spectrum of symptoms and its tendency to persist throughout a patient's lifetime, often beginning in early childhood.

In Table 2, [15] presents the attributes of the rheumatic fever dataset along with their corresponding codings. These attributes encompass:

- Sex ( $S$ ) =  $\{M, F\} = \{0, 1\}$ .
- Pharyngitis ( $F$ ) =  $\{yes, no\} = \{1, 0\}$ .
- Arthritis ( $A$ ) =  $\{0, 1, 2\}$ .
- Carditis ( $R$ ) =  $\{affected, not\ affected\} = \{1, 0\}$ .
- Chorea ( $K$ ) =  $\{yes, no\} = \{1, 0\}$ .
- ESR ( $E$ ) =  $\{normal, high\} = \{0, 1\}$ .
- Abdominal Pain ( $P$ ) =  $\{absent, present\} = \{0, 1\}$ .
- Headache ( $H$ ) =  $\{yes, no\} = \{1, 0\}$ .
- The decision attribute is Diagnosis ( $D$ ) =  $\{rheumatic\ arthritis, rheumatic\ carditis, rheumatic\ arthritis\ and\ carditis\}$ .

This study aims to leverage these attributes and their corresponding coding to explore data reduction techniques, enhancing our understanding of patterns within the complex realm of rheumatic fever symptoms and their associated diagnoses.

Table 3 outlines the distinctive features of 5 patients, depicted through 8 symptoms (attributes). These attributes play a pivotal role in ascertaining the diagnosis for each individual, encapsulated within the decision attribute.

Likewise, Table 4 provides details concerning another set of 5 patients, characterized by the identical 8 symptoms (attributes) employed for diagnosis, aligning with those delineated in Table 3. Based on this foundation, we then proceed to construct a multi-valued information system (MIS), presented in Table 5.

Table 2: The data pertaining to the disease of rheumatic fever.

Name of attribute	Attribute values	Attribute denotes to
<i>Sex (S)</i>	$s_1$	Male
	$s_2$	Female
<i>Pharyngitis (F)</i>	$f_1$	Yes
	$f_2$	No
<i>Arthritis (A)</i>	$a_0$	No arthritis
	$a_1$	Began in the knee
	$a_2$	Began in the ankle
<i>Carditis (R)</i>	$r_1$	Affected
	$r_2$	Not affected
<i>Chorea (K)</i>	$k_1$	Yes
	$k_2$	No
<i>ESR (E)</i>	$e_1$	Normal
	$e_2$	High
<i>Abdominal pain (P)</i>	$p_1$	Absent
	$p_2$	Present
<i>Headache (H)</i>	$h_1$	Yes
	$h_2$	No
<i>Diagnosis (D)</i>	$d_1$	Rheumatic arthritis
	$d_2$	Rheumatic carditis
	$d_3$	Rheumatic arthritis and carditis

Table 3: The information system related to rheumatic fever.

Attributes Patients	<i>S</i>	<i>F</i>	<i>A</i>	<i>R</i>	<i>K</i>	<i>E</i>	<i>P</i>	<i>H</i>	<i>D</i>
$x_1$	$s_2$	$f_1$	$a_1$	$r_1$	$k_1$	$e_1$	$p_1$	$h_2$	$d_3$
$x_2$	$s_1$	$f_1$	$a_1$	$r_1$	$k_1$	$e_2$	$p_1$	$h_1$	$d_3$
$x_3$	$s_2$	$f_1$	$a_2$	$r_1$	$k_2$	$e_1$	$p_1$	$h_2$	$d_3$
$x_4$	$s_1$	$f_1$	$a_1$	$r_2$	$k_2$	$e_1$	$p_1$	$h_2$	$d_1$
$x_5$	$s_1$	$f_2$	$a_0$	$r_1$	$k_2$	$e_1$	$p_2$	$h_2$	$d_2$

From the above tables, we generate the following binary relations:

1. For  $A \subseteq At$ :  $R_a = \{(x, y) : f_a(x) \subseteq f_a(y), \forall a \in C\}$ , where  $C = \{\rho, \sigma, \mu\}$  represents a set of condition attributes.
2. For  $B \subseteq At$ ,  $R_B = \{(x, y) : f_a(x) \subseteq f_a(y), \forall a \in B, B \subseteq At\}$ .

It is clear that  $R_a$  and  $R_B$  are reflexive and transitive but not symmetric.

In Table 5, where  $U$  represents the set of objects, the set of condition attributes is denoted as  $C = \{\rho, \sigma, \mu\}$ , and the decision attribute is the diagnosis  $D$ . The binary relation  $R_a$  is defined, such that  $a$  represents an element from the power set of condition attributes  $\{\rho, \sigma, \mu\}$ .

The power set of condition attributes is represented by

$$P(C) = \{C, \emptyset, \rho, \sigma, \mu, \rho\sigma, \rho\mu, \sigma\mu\}.$$

Table 4: Encoding data for rheumatic fever.

Attribute representation	Denotes to?	Attribute values	Denotes to?
$\rho$	$\{S, K\}$	$\rho_1$	S takes $s_1$
		$\rho_2$	K takes $k_1$
		$\rho_3$	Each of $\{S, K\}$ takes $\{s_2, k_2\}$
$\sigma$	$\{F, A, E\}$	$\sigma_1$	F takes $f_1$
		$\sigma_2$	A takes $a_1$
		$\sigma_3$	A takes $a_2$
		$\sigma_4$	E takes $e_1$
		$\sigma_5$	Each of $\{F, A, E\}$ takes $\{f_2, a_0, e_2\}$
$\mu$	$\{R, P, H\}$	$\mu_1$	R takes $R_1$
		$\mu_2$	P takes $p_1$
		$\mu_3$	H takes $h_1$
		$\mu_4$	Each of $\{R, P, H\}$ takes $\{R_2, p_2, h_2\}$
$D$	Diagnosis	$d_1$	Rheumatic arthritis
		$d_2$	Rheumatic carditis
		$d_3$	Rheumatic arthritis and carditis

Table 5: A multi-valued information system about data of rheumatic fever.

$x_1$	$\{\rho_2\}$	$\{\sigma_1, \sigma_2, \sigma_3\}$	$\{\mu_1, \mu_2\}$	$\{d_3\}$
$x_2$	$\{\rho_1, \rho_2\}$	$\{\sigma_1, \rho_2\}$	$\{\mu_1, \mu_2, \mu_3\}$	$\{d_3\}$
$x_3$	$\{\rho_3\}$	$\{\sigma_1, \sigma_3, \sigma_4\}$	$\{\mu_1, \mu_2\}$	$\{d_3\}$
$x_4$	$\{\rho_1\}$	$\{\sigma_1, \sigma_2, \sigma_4\}$	$\{\mu_2\}$	$\{d_1\}$
$x_5$	$\{\rho_1\}$	$\{\sigma_4\}$	$\{\mu_1\}$	$\{d_2\}$

Now we generate the next binary relations on  $U$  as follows:

$$R_{B \subseteq C} = \{(x, y) : f_{B \subseteq C}(x) \subseteq f_{B \subseteq C}(y), \forall B \subseteq C, B \neq \emptyset, \forall x, y \in U\}.$$

With the aforementioned context established, we proceed to build the subsequent relations based on the condition attributes:

$$\begin{aligned}
 R_{\{\rho\}} &= \{(x_1, x_1), (x_1, x_2), (x_2, x_2), (x_3, x_3), (x_4, x_2), (x_4, x_4), (x_4, x_5), (x_5, x_2), \\
 &\quad (x_5, x_4), (x_5, x_5)\}, \\
 R_{\{\sigma\}} &= \{(x_1, x_1), (x_1, x_4), (x_2, x_1), (x_2, x_2), (x_2, x_4), (x_3, x_3), (x_4, x_1), (x_4, x_4), \\
 &\quad (x_5, x_1), (x_5, x_3), (x_5, x_4), (x_5, x_5)\}, \\
 R_{\{\mu\}} &= \{(x_1, x_1), (x_1, x_2), (x_1, x_3), (x_2, x_2), (x_3, x_1), (x_3, x_2), (x_3, x_3), (x_4, x_1), \\
 &\quad (x_4, x_2), (x_4, x_3), (x_4, x_4), (x_5, x_1), (x_5, x_2), (x_5, x_3), (x_5, x_5)\}, \\
 R_{\{\rho, \sigma\}} &= \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4), (x_5, x_4), (x_5, x_5)\}, \\
 R_{\{\rho, \mu\}} &= \{(x_1, x_1), (x_1, x_2), (x_2, x_2), (x_3, x_3), (x_4, x_2), (x_4, x_4), (x_5, x_2), (x_5, x_5)\}, \\
 R_{\{\sigma, \mu\}} &= \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_1), (x_4, x_4), (x_5, x_1), (x_5, x_3), (x_5, x_5)\}, \\
 R_C &= \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4), (x_5, x_2), (x_5, x_5)\}.
 \end{aligned}$$

$R_{\{\rho\}}$  generates two subbases using right and left blocks of it as follows:

$$S_{\{\rho\}}^r = \{\{x_2\}, \{x_1, x_2\}, \{x_3\}, \{x_2, x_4, x_5\}\},$$

$$S_{\{\rho\}}^l = \{\{x_1\}, \{x_3\}, \{x_4, x_5\}, \{x_1, x_2, x_4, x_5\}\}.$$

Since  $S_{\{\rho\}}^{rl} = S_{\{\rho\}}^r \cup S_{\{\rho\}}^l$ , then

$$S_{\{\rho\}}^{rl} = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_4, x_5\}, \{x_2, x_4, x_5\}, \{x_1, x_2, x_4, x_5\}\}.$$

$R_{\{\sigma\}}$  generates two subbases using right and left blocks of it as follows:

$$S_{\{\sigma\}}^r = \{\{x_3\}, \{x_1, x_4\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4, x_5\}\},$$

$$S_{\{\sigma\}}^l = \{\{x_2\}, \{x_5\}, \{x_3, x_5\}, \{x_1, x_2, x_4, x_5\}\}.$$

Since  $S_{\{\sigma\}}^{rl} = S_{\{\sigma\}}^r \cup S_{\{\sigma\}}^l$ , then

$$S_{\{\sigma\}}^{rl} = \{\{x_2\}, \{x_3\}, \{x_5\}, \{x_1, x_4\}, \{x_3, x_5\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_4, x_5\}, \{x_1, x_3, x_4, x_5\}\}.$$

$R_{\{\mu\}}$  generates two subbases using right and left blocks of it as follows:

$$S_{\{\mu\}}^r = \{\{x_2\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_5\}\},$$

$$S_{\{\mu\}}^l = \{\{x_4\}, \{x_5\}, \{x_1, x_3, x_4, x_5\}, U\}.$$

Since  $S_{\{\mu\}}^{rl} = S_{\{\mu\}}^r \cup S_{\{\mu\}}^l$ , then

$$S_{\{\mu\}}^{rl} = \{\{x_2\}, \{x_4\}, \{x_5\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_5\}, \{x_1, x_3, x_4, x_5\}, U\}.$$

$R_{\{\rho, \sigma\}}$  generates two subbases using right and left blocks of it as follows:

$$S_{\{\rho, \sigma\}}^r = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_4, x_5\}\},$$

$$S_{\{\rho, \sigma\}}^l = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_5\}, \{x_4, x_5\}\}.$$

Since  $S_{\{\rho, \sigma\}}^{rl} = S_{\{\rho, \sigma\}}^r \cup S_{\{\rho, \sigma\}}^l$ , then

$$S_{\{\rho, \sigma\}}^{rl} = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_4, x_5\}\}.$$

$R_{\{\rho, \mu\}}$  generates two subbases using right and left blocks of it as follows:

$$S_{\{\rho, \mu\}}^r = \{\{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_2, x_4\}, \{x_2, x_5\}\},$$

$$S_{\{\rho, \mu\}}^l = \{\{x_1\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_1, x_2, x_4, x_5\}\}.$$

Since  $S_{\{\rho, \mu\}}^{rl} = S_{\{\rho, \mu\}}^r \cup S_{\{\rho, \mu\}}^l$ , then

$$S_{\{\rho, \mu\}}^{rl} = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_1, x_2\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_1, x_2, x_4, x_5\}\}.$$

$R_{\{\sigma, \mu\}}$  generates two subbases using right and left blocks of it as follows:

$$S_{\{\sigma, \mu\}}^r = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_4\}, \{x_1, x_3, x_5\}\},$$

$$S_{\{\sigma, \mu\}}^l = \{\{x_2\}, \{x_4\}, \{x_5\}, \{x_3, x_5\}, \{x_1, x_4, x_5\}\}.$$

Since  $S_{\{\sigma, \mu\}}^{rl} = S_{\{\sigma, \mu\}}^r \cup S_{\{\sigma, \mu\}}^l$ , then

$$S_{\{\sigma, \mu\}}^{rl} = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}, \{x_1, x_4\}, \{x_3, x_5\}, \{x_1, x_3, x_5\}, \{x_1, x_4, x_5\}\}.$$

$R_C$  generates two subbases using right and left blocks of it as follows:

$$S_C^r = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\},$$

$$S_C^l = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\}.$$

Since  $S_{\{\rho, \mu\}}^{rl} = S_C^r \cup S_C^l$ , then

$$S_C^{rl} = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_5\}\}.$$

Thus, we have the following tritopologies:

$$\begin{aligned} Tri \tau_{\{\rho\}} &= \{U, \emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_4, x_5\}, \\ &\quad \{x_1, x_4, x_5\}, \{x_2, x_4, x_5\}, \{x_3, x_4, x_5\}, \{x_1, x_2, x_4, x_5\}, \\ &\quad \{x_1, x_3, x_4, x_5\}, \{x_2, x_3, x_4, x_5\}\}, \\ Tri \tau_{\{\sigma\}} &= \{U, \emptyset, \{x_2\}, \{x_3\}, \{x_5\}, \{x_1, x_4\}, \{x_2, x_3\}, \{x_2, x_5\}, \{x_3, x_5\}, \\ &\quad \{x_1, x_2, x_4\}, \{x_2, x_3, x_5\}, \{x_1, x_4, x_5\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_3, x_4\}, \\ &\quad \{x_1, x_2, x_4, x_5\}, \{x_1, x_3, x_4, x_5\}\}, \\ Tri \tau_{\{\mu\}} &= \{U, \emptyset, \{x_2\}, \{x_4\}, \{x_5\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_4, x_5\}, \{x_1, x_2, x_3\}, \\ &\quad \{x_2, x_4, x_5\}, \{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_5\}, \{x_1, x_3, x_4, x_5\}\}, \\ Tri \tau_{\{\rho, \sigma\}} &= P(U). \\ Tri \tau_{\{\rho, \mu\}} &= P(U). \\ Tri \tau_{\{\sigma, \mu\}} &= P(U). \\ Tri \tau_C &= P(U). \\ Tri \tau_D &= \{U, \emptyset, \{x_4\}, \{x_5\}, \{x_4, x_5\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_5\}\}. \end{aligned}$$

These tritopologies serve as a fundamental knowledge base for our system, allowing us to derive decision rules from the data provided in Table 5.

The distinguishable subsets of the decision attribute are as follows:

$$\begin{aligned} D_1 &= \text{Decision (Rheumatic arthritis)} = \{x_4\}. \\ D_2 &= \text{Decision (Rheumatic carditis)} = \{x_5\}. \\ D_3 &= \text{Decision (Rheumatic arthritis and carditis)} = \{x_1, x_2, x_3\}. \end{aligned}$$

Now we calculate the *tri*-lower approximation, *tri*-upper approximation, *tri*- $\delta\beta$ -lower approximation, *tri*- $\delta\beta$ -upper approximation of the subset  $D_i$ ,  $i = 1, 2, 3$  are given by:

For the attribute  $\rho$ , we get:

$$\begin{aligned} \underline{Tri \tau^{\{\rho\}}}(D_1) &= \underline{Tri \tau^{\{\rho\}}}(D_2) = \emptyset, \\ \underline{Tri \tau^{\{\rho\}}}(D_3) &= \{x_1, x_2, x_3\}, \\ \overline{Tri \tau^{\{\rho\}}}(D_1) &= \{x_4, x_5\}, \\ \overline{Tri \tau^{\{\rho\}}}(D_2) &= \{x_4, x_5\}, \\ \overline{Tri \tau^{\{\rho\}}}(D_3) &= \{x_1, x_2, x_3\}. \end{aligned}$$

As  $Tri \tau^{\{\rho\}} BON(D_3)$  results in an empty set, it follows that the decision “**Rheumatic arthritis and carditis**” is characterized by a precise 100% certainty, indicating that its boundary region

remains devoid of elements.

$$\begin{aligned} \overline{Tri \tau_{\delta\beta}^{\{\rho\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho\}}}(D_3) &= \{x_1, x_2, x_3\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho\}}}(D_3) &= \{x_1, x_2, x_3\}. \end{aligned}$$

Since  $Tri \tau_{\delta\beta}^{\{\rho\}} BON(D_1) = \emptyset$ ,  $Tri \tau_{\delta\beta}^{\{\rho\}} BON(D_2) = \emptyset$ , and  $Tri \tau_{\delta\beta}^{\{\rho\}} BON(D_3) = \emptyset$ , it can be concluded that the decisions “**Rheumatic arthritis**, **Rheumatic carditis**, and **Rheumatic arthritis and carditis**” hold a precise 100% certainty, signifying that their respective boundary regions remain unpopulated.

For the attribute  $\sigma$ , we get:

$$\begin{aligned} \overline{Tri \tau^{\{\sigma\}}}(D_1) &= \emptyset, \\ \overline{Tri \tau^{\{\sigma\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau^{\{\sigma\}}}(D_3) &= \{x_2, x_3\}, \\ \overline{Tri \tau^{\{\sigma\}}}(D_1) &= \{x_1, x_4\}, \\ \overline{Tri \tau^{\{\sigma\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau^{\{\sigma\}}}(D_3) &= \{x_1, x_2, x_3, x_4\}. \end{aligned}$$

As  $Tri \tau^{\{\sigma\}} BON(D_2)$  results in an empty set, it follows that the decision “**Rheumatic carditis**” attains a precise accuracy of 100%, with its boundary region being devoid of elements.

$$\begin{aligned} \overline{Tri \tau_{\delta\beta}^{\{\sigma\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\sigma\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\sigma\}}}(D_3) &= \{x_1, x_2, x_3\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\sigma\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\sigma\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\sigma\}}}(D_3) &= \{x_1, x_2, x_3\}. \end{aligned}$$

Given that  $Tri \tau_{\delta\beta}^{\{\sigma\}} BON(D_1)$  results in an empty set along with  $Tri \tau_{\delta\beta}^{\{\sigma\}} BON(D_2)$  and  $Tri \tau_{\delta\beta}^{\{\sigma\}} BON(D_3)$ , it follows that the decisions “**Rheumatic arthritis**, **Rheumatic carditis**, and **Rheumatic arthritis and carditis**” all achieve a precise accuracy of 100%, with their respective boundary regions being void of elements.

For the attribute  $\mu$ , we get:

$$\begin{aligned} \overline{Tri \tau^{\{\mu\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau^{\{\mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau^{\{\mu\}}}(D_3) &= \{x_1, x_2, x_3\}, \\ \overline{Tri \tau^{\{\mu\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau^{\{\mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau^{\{\mu\}}}(D_3) &= \{x_1, x_2, x_3\}. \end{aligned}$$

Given that  $Tri \tau^{\{\mu\}} BON(D_1)$  results in an empty set along with  $Tri \tau^{\{\mu\}} BON(D_2)$  and  $Tri \tau^{\{\mu\}} BON(D_3)$ , it follows that the decisions “**Rheumatic arthritis, Rheumatic carditis, and Rheumatic arthritis and carditis**” all achieve a precise accuracy of 100%, with their respective boundary regions being void of elements.

$$\begin{aligned} \overline{Tri \tau_{\delta\beta}^{\{\mu\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\mu\}}}(D_3) &= \{x_1, x_2, x_3\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\mu\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\mu\}}}(D_3) &= \{x_1, x_2, x_3\}. \end{aligned}$$

Given that  $Tri \tau_{\delta\beta}^{\{\mu\}} BON(D_1)$  results in an empty set along with  $Tri \tau_{\delta\beta}^{\{\mu\}} BON(D_2)$  and  $Tri \tau_{\delta\beta}^{\{\mu\}} BON(D_3)$ , it follows that the decisions “**Rheumatic arthritis, Rheumatic carditis, and Rheumatic arthritis and carditis**” all achieve a precise accuracy of 100%, with their respective boundary regions being void of elements.

For the attributes  $\rho$  and  $\sigma$ , we get:

$$\begin{aligned} \overline{Tri \tau^{\{\rho,\sigma\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau^{\{\rho,\mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau^{\{\mu\}}}(D_3) &= \{x_1, x_2, x_3\}, \\ \overline{Tri \tau^{\{\rho,\sigma\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau^{\{\rho,\sigma\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau^{\{\rho,\sigma\}}}(D_3) &= \{x_1, x_2, x_3\}. \end{aligned}$$

Given that  $Tri \tau^{\{\rho,\sigma\}} BON(D_1)$  results in an empty set along with  $Tri \tau^{\{\rho,\sigma\}} BON(D_2)$  and  $Tri \tau^{\{\rho,\sigma\}} BON(D_3)$ , it follows that the decisions “**Rheumatic arthritis, Rheumatic carditis, and Rheumatic arthritis and carditis**” all achieve a precise accuracy of 100%, with their respective

boundary regions being void of elements.

$$\begin{aligned} \overline{Tri \tau_{\delta\beta}^{\{\rho,\sigma\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\sigma\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\sigma\}}}(D_3) &= \{x_1, x_2, x_3\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\sigma\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\sigma\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\sigma\}}}(D_3) &= \{x_1, x_2, x_3\}. \end{aligned}$$

Given that  $Tri \tau_{\delta\beta}^{\{\rho,\sigma\}} BON(D_1)$ ,  $Tri \tau_{\delta\beta}^{\{\rho,\sigma\}} BON(D_2)$ , and  $Tri \tau_{\delta\beta}^{\{\rho,\sigma\}} BON(D_3)$  all result in empty sets, it can be deduced that the decisions “**Rheumatic arthritis, Rheumatic carditis, and Rheumatic arthritis and carditis**” are all determined with an exact accuracy of 100%. Furthermore, this accuracy is achieved while maintaining empty boundary regions for these decisions.

For the attributes  $\rho$  and  $\mu$ , we get:

$$\begin{aligned} \overline{Tri \tau^{\{\rho,\mu\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau^{\{\rho,\mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau^{\{\rho,\mu\}}}(D_3) &= \{x_1, x_2, x_3\}, \\ \overline{Tri \tau^{\{\rho,\mu\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau^{\{\rho,\mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau^{\{\rho,\mu\}}}(D_3) &= \{x_1, x_2, x_3\}. \end{aligned}$$

Given that  $Tri \tau^{\{\rho,\mu\}} BON(D_1)$ ,  $Tri \tau^{\{\rho,\mu\}} BON(D_2)$ , and  $Tri \tau^{\{\rho,\mu\}} BON(D_3)$  all result in empty sets, it can be deduced that the decisions “**Rheumatic arthritis, Rheumatic carditis, and Rheumatic arthritis and carditis**” are all determined with an exact accuracy of 100%. Furthermore, this accuracy is achieved while maintaining empty boundary regions for these decisions.

$$\begin{aligned} \overline{Tri \tau_{\delta\beta}^{\{\rho,\mu\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\mu\}}}(D_3) &= \{x_1, x_2, x_3\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\mu\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\mu\}}}(D_3) &= \{x_1, x_2, x_3\}. \end{aligned}$$

Given that  $Tri \tau_{\delta\beta}^{\{\rho,\mu\}} BON(D_1)$ ,  $Tri \tau_{\delta\beta}^{\{\rho,\mu\}} BON(D_2)$ , and  $Tri \tau_{\delta\beta}^{\{\rho,\mu\}} BON(D_3)$  all yield empty sets, it can be concluded that the decisions “**Rheumatic arthritis, Rheumatic carditis, and Rheumatic arthritis and carditis**” each possess a precise accuracy of 100%. Furthermore, these decisions maintain empty boundary regions.

For the attributes  $\sigma$  and  $\mu$ , we get:

$$\begin{aligned} \underline{Tri \tau^{\{\sigma, \mu\}}}(D_1) &= \{x_4\}, \\ \underline{Tri \tau^{\{\rho, \mu\}}}(D_2) &= \{x_5\}, \\ \underline{Tri \tau^{\{\sigma, \mu\}}}(D_3) &= \{x_1, x_2, x_3\}, \\ \overline{Tri \tau^{\{\sigma, \mu\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau^{\{\sigma, \mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau^{\{\sigma, \mu\}}}(D_3) &= \{x_1, x_2, x_3\}. \end{aligned}$$

Given that  $Tri \tau^{\{\sigma, \mu\}} BON(D_1)$ ,  $Tri \tau^{\{\sigma, \mu\}} BON(D_2)$ , and  $Tri \tau^{\{\sigma, \mu\}} BON(D_3)$  all yield empty sets, it can be concluded that the decisions “**Rheumatic arthritis, Rheumatic carditis, and Rheumatic arthritis and carditis**” each possess a precise accuracy of 100%. Furthermore, these decisions maintain empty boundary regions.

$$\begin{aligned} \underline{Tri \tau_{\delta\beta}^{\{\sigma, \mu\}}}(D_1) &= \{x_4\}, \\ \underline{Tri \tau_{\delta\beta}^{\{\sigma, \mu\}}}(D_2) &= \{x_5\}, \\ \underline{Tri \tau_{\delta\beta}^{\{\sigma, \mu\}}}(D_3) &= \{x_1, x_2, x_3\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\sigma, \mu\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\sigma, \mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\sigma, \mu\}}}(D_3) &= \{x_1, x_2, x_3\}. \end{aligned}$$

Given that  $Tri \tau_{\delta\beta}^{\{\sigma, \mu\}} BON(D_1)$ ,  $Tri \tau_{\delta\beta}^{\{\sigma, \mu\}} BON(D_2)$ , and  $Tri \tau_{\delta\beta}^{\{\sigma, \mu\}} BON(D_3)$  all yield empty sets, it can be concluded that the decisions “**Rheumatic arthritis, Rheumatic carditis, and Rheumatic arthritis and carditis**” each possess a precise accuracy of 100%. Furthermore, these decisions maintain empty boundary regions.

For the attributes  $\rho, \sigma$  and  $\mu$ , we get:

$$\begin{aligned} \underline{Tri \tau^{\{\rho, \sigma, \mu\}}}(D_1) &= \{x_4\}, \\ \underline{Tri \tau^{\{\rho, \sigma, \mu\}}}(D_2) &= \{x_5\}, \\ \underline{Tri \tau^{\{\rho, \sigma, \mu\}}}(D_3) &= \{x_1, x_2, x_3\}, \\ \overline{Tri \tau^{\{\rho, \sigma, \mu\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau^{\{\rho, \sigma, \mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau^{\{\rho, \sigma, \mu\}}}(D_3) &= \{x_1, x_2, x_3\}. \end{aligned}$$

Given that  $Tri \tau^{\{\rho, \sigma, \mu\}} BON(D_1)$ ,  $Tri \tau^{\{\rho, \sigma, \mu\}} BON(D_2)$ , and  $Tri \tau^{\{\rho, \sigma, \mu\}} BON(D_3)$  all yield empty sets, it can be concluded that the decisions “**Rheumatic arthritis, Rheumatic carditis, and Rheumatic arthritis and carditis**” each possess a precise accuracy of 100%. Furthermore, these

decisions maintain empty boundary regions.

$$\begin{aligned} \overline{Tri \tau_{\delta\beta}^{\{\rho,\sigma,\mu\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\sigma,\mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\sigma,\mu\}}}(D_3) &= \{x_1, x_2, x_3\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\sigma,\mu\}}}(D_1) &= \{x_4\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\sigma,\mu\}}}(D_2) &= \{x_5\}, \\ \overline{Tri \tau_{\delta\beta}^{\{\rho,\sigma,\mu\}}}(D_3) &= \{x_1, x_2, x_3\}. \end{aligned}$$

Given that  $Tri \tau_{\delta\beta}^{\{\rho,\sigma,\mu\}} BON(D_1)$ ,  $Tri \tau_{\delta\beta}^{\{\rho,\sigma,\mu\}} BON(D_2)$ , and  $Tri \tau_{\delta\beta}^{\{\rho,\sigma,\mu\}} BON(D_3)$  all yield empty sets, it can be concluded that the decisions “**Rheumatic arthritis, Rheumatic carditis, and Rheumatic arthritis and carditis**” each possess a precise accuracy of 100%. Furthermore, these decisions maintain empty boundary regions.

## 6 Conclusion and Discussion

In this study, we have expanded upon Pawlak’s approximation space by introducing a novel family of binary relations, leading to the creation of a tritopological approximation space. By leveraging right neighborhoods and left neighborhoods derived from these relations, we have formulated three distinct topologies. These topologies have enabled the definition of *tri*-lower and *tri*-upper approximations for any subset within the universe. Our comprehensive analysis has demonstrated that the  $\delta\beta$ -model outperformed other proposed models, significantly reducing the *tri*- $\delta\beta$ -boundary region of vague concepts. Consequently, this model provides a promising avenue for enhancing the precision of vague concepts. Thus, the *tri*- $\delta\beta$  approach to rough set approximations holds great potential for addressing vagueness in a wide range of real-life applications.

In future work, we intend to further develop the theoretical framework within the context of *tri*-concepts. Specifically, we plan to investigate additional axiomatic properties of *tri*-approximation space and explore fundamental topological concepts in *tri*-spaces, such as separation axioms, compactness, and connectedness. The *tri*-hypothesis holds promise for advanced studies and real-life applications in engineering problems. Additionally, our future research will focus on extending the application of the developed framework to fields beyond tritopological approximation space, such as fuzzy set theory and soft set approaches. By continuing to advance this line of inquiry, we aim to contribute to the broader understanding and utilization of tritopological methods in solving complex, real-world problems.

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**Conflicts of Interest** The authors assert that they have no conflicts of interest to disclose in relation to the current study.

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